

# HOCHSTER'S SMALL MCM CONJECTURE FOR THREE-DIMENSIONAL WEAKLY F-SPLIT RINGS

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**ABSTRACT.** We prove Hochster's small MCM conjecture for three-dimensional complete F-pure rings. We deduce this from a more general criterion, and show that only a weakening of the notion of F-purity is needed, to wit, being weakly F-split. We conjecture that any complete ring is weakly F-split.

## 1. INTRODUCTION

Let  $R$  be a  $d$ -dimensional Noetherian local ring with residue field  $k$ . A module  $M$  is called a *small MCM* (=finitely generated maximal Cohen-Macaulay module), if it has depth  $d$ . Since all modules will tacitly be assumed to be finitely generated, we drop the modifier 'small' altogether. Without any further assumption on  $R$ , these might not exist, but Hochster conjectured that any complete local ring admits an MCM. The first unknown case of Hochster's conjecture is in dimension three. We proved in [8] that if MCM's exist such that their multiplicities are not too big, then the equal characteristic zero case follows from the positive characteristic case; in the mixed characteristic case, we even know less. In this paper, we will mainly tackle the case that  $R$  is complete, of positive characteristic  $p$ , and has dimension  $d = 3$ . I now briefly describe the strategy to obtain a small MCM in this setting.

**1.1. Higher pseudo-canonical modules.** If  $(R, \mathfrak{m})$  is complete and Cohen-Macaulay, it admits a canonical module  $\omega$ . Without the Cohen-Macaulay assumption, we lack Grothendieck vanishing, forcing us to define a sequence of modules instead: given a module  $M$ , we define its  $i$ -th *higher pseudo-canonical module*  $\mathbf{K}_i(M)$  as the Matlis dual of the local cohomology module  $H_{\mathfrak{m}}^{d-i}(M)$ . Of course, if  $M$  is MCM, then all higher pseudo-canonical modules  $\mathbf{K}_i(M)$ , for  $i = 1, \dots, d$ , vanish. Our first main result (without any assumption on the characteristic) is a substantial weakening of this:

**1.2. Theorem.** *A three-dimensional complete local ring admits an MCM if and only if there exists a module  $M$  such that  $\mathbf{K}_1(M)$  has positive depth.*

In fact, if  $M$  satisfies the above assumption, then  $\mathbf{K}_0(M)$  is an MCM. We turn now to characteristic  $p > 0$  to construct such modules  $M$ .

**1.3. The Frobenius transform of a module.** Let  $\mathbf{F}_q$  be the Frobenius morphism  $x \mapsto x^q$ , where  $q = p^n$  is some power of the characteristic (when  $q$  is clear, we drop reference to it). Given an  $R$ -module  $M$ , we denote its pull-back via  $\mathbf{F}_q$  by  $\mathbf{F}_{q*}M$  or just  $\mathbf{F}_*M$  and call it the *Frobenius transform* of  $M$ ; its elements are denoted by  $*m$ , and its  $R$ -module structure is then given by  $r*m := *r^q m$ . If  $k$  is perfect,

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then  $R$  is  $F$ -finite, and hence every Frobenius transform  $\mathbf{F}_*M$  is again finitely generated. If we endow  $\mathbf{F}_*R$  with the multiplication it inherits from  $R$ , namely  $(*a) \cdot (*b) := *ab$ , then it becomes an  $R$ -algebra, and its structure homomorphism is just another incarnation of the Frobenius homomorphism. Our second result, in arbitrary dimension, is:

**1.4. Proposition.** *If there exists a direct sum decomposition  $\mathbf{F}_*Q \cong Q \oplus M$ , for some  $R$ -modules  $Q$  and  $M$ , then  $M$  satisfies the condition in Theorem 1.2, that is to say,  $\mathbf{K}_1(M)$  has positive depth.*

Recall that  $R$  is called  $F$ -pure if  $\mathbf{F}_q$  (whence  $\mathbf{F}_p$ ) is pure, which in the complete case, is equivalent with  $R$  being a direct summand of  $\mathbf{F}_*R$ .<sup>1</sup> We substantially weaken this condition by calling  $R$  *weakly  $F$ -split*, if it is  $F$ -finite and there exists some module  $Q$  which is a direct summand of its Frobenius transform  $\mathbf{F}_*Q$  (we will then call such a  $Q$   *$F$ -split*; but be aware that there is in general no longer a morphism from  $Q$  to its Frobenius transform  $\mathbf{F}_*Q$ ). So we proved (note that there are plenty of non-Cohen-Macaulay  $F$ -pure rings):

**Main Theorem.** *In dimension three, any complete weakly  $F$ -split ring, whence in particular any complete  $F$ -pure ring, admits a small MCM.*  $\square$

**1.5. Proof of Proposition 1.4.** The proof requires two ingredients. Firstly, we prove (§4.7) that in general, we have an isomorphism

$$(1) \quad \mathbf{F}_*(\mathbf{K}_i(M)) \cong \mathbf{K}_i(\mathbf{F}_*M).$$

Note, however, that this is a non-canonical isomorphism, which therefore requires some work. The second ingredient is the behavior of ordinal length under Frobenius transforms. However, for the case at hand, we do not need to rely on ordinal length (a transfinite version of ordinary length studied in [9]), and so we can make the following ad hoc observations: let  $\ell_0(M)$  be the length of zero-th local cohomology module  $H_m^0(M)$ . In particular,  $M$  has positive depth if and only if  $\ell_0(M) = 0$ . We show (Corollary 3.6) that if  $k$  is perfect, then

$$(2) \quad \ell_0(M) = \ell_0(\mathbf{F}_*M).$$

Now define  $h(M) := \ell_0(\mathbf{K}_1(M))$ , so that the condition in Theorem 1.2 is equivalent with  $h(M) = 0$ . Suppose now that  $R$  is weakly  $F$ -split, witnessed by an  $F$ -split module  $Q$ , that is to say, a decomposition  $\mathbf{F}_*Q \cong Q \oplus M$ . Since depth is preserved under faithfully flat descent, we can make a base change (a scalar extension  $R \rightarrow R_{k^{1/p^\infty}}^\wedge$ , with  $k^{1/p^\infty}$  the perfect hull of the residue field  $k$ , in the sense of [7, §3]) so that the residue field becomes perfect. Since  $h$  is additive on direct sums, we get  $h(\mathbf{F}_*Q) = h(Q) + h(M)$ . On the other hand, we have

$$(3) \quad h(\mathbf{F}_*Q) = \ell_0(\mathbf{K}_1(\mathbf{F}_*Q)) \stackrel{(1)}{=} \ell_0(\mathbf{F}_*(\mathbf{K}_1(Q))) \stackrel{(2)}{=} \ell_0(\mathbf{K}_1(Q)) = h(Q)$$

from which it follows that  $h(M) = 0$ .  $\square$

We will prove Theorem 1.2 in §2, whereas sections §3 and §4 are then devoted to proving respectively (2) and (1), completing the proof of our main theorem. In §5, we give a few examples (without proof) of  $F$ -split modules. However, as

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<sup>1</sup>I am grateful to Florian Enescu for providing me the following argument for the converse: suppose  $s: R \rightarrow \mathbf{F}_*R$  is the embedding as a direct summand, and let  $s(1) := *c$ ; then  $s$  is the composition of the (Frobenius) homomorphism  $R \rightarrow \mathbf{F}_*R$  followed by multiplication with  $*c$ , and as  $s$  is split, so is then the former homomorphism, which is the definition of being  $F$ -split.

calculations blow up fast, we can only give examples over hypersurfaces, which are of course already Cohen-Macaulay. We should point out that our use of Frobenius to obtain small MCM's is different from the graded case as explained in [4] (for a proof, see [5]), and also different from the toric case [8].

## 2. HIGHER PSEUDO-CANONICAL MODULES

In this section, we fix a complete local ring  $(R, \mathfrak{m})$  of dimension  $d$ . We also need to work occasionally with non-finitely generated (aka *big*) modules and to emphasize this, we will denote them by capital Greek letters. Let  $k$  be the residue field of  $R$  and  $E$  the injective hull of  $k$ . Recall that the *Matlis dual* of a (big)  $R$ -module  $\Omega$  is given by

$$\Omega^\vee := \operatorname{Hom}_R(\Omega, E)$$

It is an exact, contravariant functor, sending Noetherian modules to Artinian ones, and vice versa, and these are then canonically isomorphic to their biduals (see, for instance, [6, §18]). In particular, the Matlis dual of a module of finite length has again finite length, and in fact, the same length.

**2.1. Definition.** Given a (finitely generated)  $R$ -module  $M$ , let us define its  $i$ -th *pseudo-canonical module*, for  $i = 0, \dots, d$ , by the rule

$$\mathbf{K}_i(M) := H_{\mathfrak{m}}^{d-i}(M)^\vee.$$

If we want to emphasize the base ring  $R$ , we will denote these modules by  $\mathbf{K}_i^R(M)$ . If  $i = 0$ , we just write  $\mathbf{K}(M)$  for  $\mathbf{K}_0(M)$ . Since each  $H_{\mathfrak{m}}^\bullet(M)$  is Artinian, the  $\mathbf{K}_i(M)$  are all finitely generated  $R$ -modules. By Grothendieck vanishing, if  $s$  and  $r$  are the respective depth and dimension of  $M$ , then  $d - r$  and  $d - s$  are respectively the smallest and largest index  $i$  for which  $\mathbf{K}_i(M) \neq 0$ . In particular,  $M$  is an MCM if and only if  $\mathbf{K}_i(M) = 0$ , for all  $i \geq 1$ . The following fact will help us in studying pseudo-canonical modules, as it often allows us to reduce to the Cohen-Macaulay case.

**2.2. Proposition.** *Let  $\varphi: (S, \mathfrak{n}) \rightarrow (R, \mathfrak{m})$  be a finite morphism of complete local rings of relative dimension  $r := \dim(S) - \dim(R)$ . Let  $\Omega$  be an  $R$ -module, and let  $\varphi_*\Omega$  be its pull-back along  $\varphi$ , that is to say, viewed as an  $S$ -module.*

- (2.2.a) *The Matlis  $R$ -dual of  $\Omega$  is isomorphic over  $S$  to the Matlis  $S$ -dual of  $\varphi_*\Omega$ , and  $H_{\mathfrak{m}}^i(\Omega) \cong H_{\mathfrak{n}}^i(\varphi_*\Omega)$  as  $S$ -modules, for all  $i$ .*
- (2.2.b) *For all  $i$ , we have an isomorphism of  $S$ -modules  $\mathbf{K}_i^R(\Omega) \cong \mathbf{K}_{r+i}^S(\varphi_*\Omega)$ .*
- (2.2.c) *There always exists at least one local Gorenstein (whence Cohen-Macaulay) ring  $S$  and a finite morphism  $\varphi: S \rightarrow R$  of relative dimension zero, and hence in this situation we have isomorphisms of  $S$ -modules*

$$\mathbf{K}_i^R(\Omega) \cong \mathbf{K}_i^S(\varphi_*\Omega).$$

*Proof.* Let  $E_S$  and  $E_R$  be the injective hull of the respective residue fields of  $S$  and  $R$ , and let  $\Omega_R^\vee := \operatorname{Hom}_R(\Omega, E_R)$  and  $\Omega_S^\vee := \operatorname{Hom}_S(\varphi_*\Omega, E_S)$  be the respective Matlis duals of  $\Omega$  and its pull-back. Since  $\operatorname{Hom}_S(R, E_S)$  is injective as an  $R$ -module by [1, Lemma 3.1.6] and is supported only at  $\mathfrak{m}$ , it must be a power of  $E_R$ , say  $\operatorname{Hom}_S(R, E_S) \cong E_R^e$ , for some  $e \in \mathbb{N}$ . Hence we have isomorphisms of  $R$ -modules

$$(\Omega_R^\vee)^e = \operatorname{Hom}_R(\Omega, E_R)^e \cong \operatorname{Hom}_R(\Omega, \operatorname{Hom}_S(R, E_S)) \cong \operatorname{Hom}_S(\varphi_*\Omega, E_S) = \Omega_S^\vee.$$

So remains to show that  $e = 1$ . Let  $k$  be the residue field of  $R$  and let  $n$  be the length of its pull-back  $\varphi_*k$  viewed as an  $S$ -module. As an  $S$ -module, the Matlis

dual  $k_S^\vee$  has the same length  $n$ , and is isomorphic to  $(k_R^\vee)^e$  by the above. Since  $k \cong k_R^\vee$ , the length as an  $S$ -module of  $\varphi_*(k_R^\vee)^e$  is equal to  $en$ , so that  $n = en$ , whence  $e = 1$ .

The analogue statement for local cohomology is well-known (see, for instance, [1, §3.5(3)]), so that combining these two facts proves (2.2.b). As for the last assertion, by Cohen's structure theorems, we can find a regular local ring  $T$  such that  $R \cong T/I$  for some height  $r$  ideal  $I \subseteq T$ . Since  $T$  is Cohen-Macaulay, we can find a regular sequence  $(x_1, \dots, x_h)$  inside  $I$ . In particular,  $S := T/(x_1, \dots, x_h)T$  is Gorenstein and  $S \rightarrow R$  has relative dimension zero.  $\square$

If  $R$  is Cohen-Macaulay, then  $\mathbf{K}(R)$  is just its canonical module by Grothendieck duality. Without the Cohen-Macaulay assumption, we only have

$$(4) \quad \mathbf{K}(M) \cong \operatorname{Hom}(M, \mathbf{K}(R)),$$

which follows from applying Matlis duality to the isomorphism  $M \otimes H_m^d(R) \cong H_m^d(M)$ . If  $S \rightarrow R$  is as in (2.2.c), then  $\mathbf{K}(R) \cong \operatorname{Hom}_S(R, S)$ , since  $S$  is then its own canonical module. Nonetheless, the pseudo-canonical modules still form a contravariant, additive  $\delta$ -functor in the sense of Grothendieck [3], that is to say:

**2.3. Proposition.** *Given an exact sequence  $0 \rightarrow N \rightarrow M \rightarrow Q \rightarrow 0$ , we get canonically defined transition maps  $\partial_i: \mathbf{K}_i(N) \rightarrow \mathbf{K}_{i+1}(Q)$  that fit into a long exact sequence*

$$\begin{aligned} 0 \rightarrow \mathbf{K}_0(Q) \rightarrow \mathbf{K}_0(M) \rightarrow \mathbf{K}_0(N) \xrightarrow{\partial_0} \mathbf{K}_1(Q) \rightarrow \mathbf{K}_1(M) \rightarrow \dots \\ \dots \rightarrow \mathbf{K}_{d-1}(N) \xrightarrow{\partial_{d-1}} \mathbf{K}_d(Q) \rightarrow \mathbf{K}_d(M) \rightarrow \mathbf{K}_d(N) \rightarrow 0. \end{aligned}$$

*Proof.* Immediate from the long exact sequence of local cohomology and the exactness of Matlis duality.  $\square$

**2.4. Remark.** Note, however, that the  $\mathbf{K}_\bullet(\cdot)$  are not derived functors if  $R$  is not Cohen-Macaulay. Put differently, we do no longer have Grothendieck duality: in general  $\mathbf{K}_i(M)$  is different from  $\operatorname{Ext}_R^i(M, \mathbf{K}(R))$ ; see Proposition 2.5 below.

Let us call an element  $m$  in a module  $M$  *small*, if  $\dim(R/\operatorname{Ann}(m))$ , that is to say the dimension of the module generated by  $m$ , is strictly smaller than the dimension of  $M$  itself. The subset of all small elements forms a submodule, denoted  $\mathfrak{s}(M)$ , and the resulting quotient  $M^{\operatorname{unm}} := M/\mathfrak{s}(M)$  is called the *unmixed quotient* of  $M$ . Let us call a module *unmixed*, if  $\mathfrak{s}(M) = 0$ . It is easy to see that  $M^{\operatorname{unm}}$  is unmixed, and in fact, it is the largest unmixed quotient of  $M$ .

**2.5. Proposition.** *Let  $d := \dim(R)$  and  $M$  a  $d$ -dimensional module. Then  $\mathbf{K}(M)$  is unmixed of dimension  $d$  and  $\dim(\mathbf{K}_i(M)) \leq d - i$ , for all  $i \geq 1$ . Moreover, if  $S \rightarrow R$  is finite of relative dimension zero and  $S$  is Cohen-Macaulay, then we have natural isomorphisms of  $S$ -modules*

$$(5) \quad \mathbf{K}_i(M) \cong \operatorname{Ext}_S^i(M, \mathbf{K}(R))$$

for all  $i$ , where for simplicity, we also wrote  $M$  for its pull-back as an  $S$ -module.

*Proof.* By (2.2.b), since the relative dimension is zero, we may view the  $\mathbf{K}_i(M)$  as the pseudo-canonical  $S$ -modules of the pull-back (justifying reference omission to the base ring). In particular, as such,  $\mathbf{K}(R)$  is just the canonical module of  $S$ , and the isomorphisms (5) are now just Grothendieck duality over  $S$  (see also [1,

Exercise 3.5.14]. Moreover, by (2.2.c), a morphism  $S \rightarrow R$  as above always exist. It is well-known ([1, Corollary 3.5.11]) that  $\text{Ext}_S^i(M, \mathbf{K}(R))$  has dimension at most  $d - i$ , whence our last claim (note that the dimension of a module does not depend on the base ring). To prove the first assertion, by (4), we only need to show that  $\mathbf{K}(R)$  is unmixed as an  $R$ -module, but this is the same as being unmixed as an  $S$ -module, which follows since the latter is the canonical module of  $S$ .  $\square$

**2.6. Lemma.** *For any  $R$ -module  $M$ , we have a canonical isomorphism  $\mathbf{K}(M) \cong \mathbf{K}(M^{\text{unm}})$  and a natural embedding  $\mathbf{K}_1(M^{\text{unm}}) \subseteq \mathbf{K}_1(M)$ .*

*Proof.* By Proposition 2.3, the exact sequence  $0 \rightarrow \mathfrak{s}(M) \rightarrow M \rightarrow M^{\text{unm}} \rightarrow 0$  gives rise to a long exact sequence

$$0 \rightarrow \mathbf{K}(M^{\text{unm}}) \rightarrow \mathbf{K}(M) \rightarrow \mathbf{K}(\mathfrak{s}(M)) \rightarrow \mathbf{K}_1(M^{\text{unm}}) \rightarrow \mathbf{K}_1(M)$$

By Grothendieck vanishing,  $\mathbf{K}(\mathfrak{s}(M)) = 0$ , and the assertion follows.  $\square$

Our first application is an abundant source of MCM's in dimension two:

**2.7. Corollary.** *If  $R$  is a two-dimensional complete local ring and  $M$  any two-dimensional  $R$ -module, then  $\mathbf{K}(M)$  is an MCM.*

*Proof.* Put  $Q := \mathbf{K}(M)$ , and since this is isomorphic to  $\mathbf{K}(M^{\text{unm}})$  by Lemma 2.6, we may already assume from the start that  $M$  is unmixed. Let  $x$  be a parameter (=element outside all maximal dimensional prime ideals) on  $R$ . By unmixedness, it is  $M$ -regular. Put  $\bar{M} := M/xM$ . The exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow \bar{M} \rightarrow 0$  yields, by Proposition 2.3 and Grothendieck vanishing, a long exact sequence

$$0 = \mathbf{K}_0(\bar{M}) \rightarrow Q \xrightarrow{x} Q \rightarrow \mathbf{K}_1(\bar{M}) \rightarrow \dots$$

so that in particular,  $Q/xQ \subseteq \mathbf{K}_1(\bar{M})$ . If we let  $\bar{R} := R/xR$ , then by (2.2.b), we have  $\mathbf{K}_1(\bar{M}) = \mathbf{K}_1^{\bar{R}}(\bar{M}) = \mathbf{K}_0^{\bar{R}}(\bar{M})$ . By Proposition 2.5, the latter is unmixed (as an  $\bar{R}$ -module), whence has depth at least one. Hence  $Q/xQ$ , being a submodule, has also depth at least one. Since  $x$  is  $Q$ -regular, as  $Q$  is unmixed of dimension two by Proposition 2.5, we showed that  $Q$  has depth at least two, i.e., is MCM.  $\square$

**2.8. Remark.** We actually proved that if  $\dim(M) = \dim(R) \geq 2$ , then  $\mathbf{K}(M)$  has depth at least two.

**2.9. Proof of Theorem 1.2.** If  $M$  is an MCM, then  $\mathbf{K}_1(M) = 0$  by Grothendieck vanishing (and the zero module has by definition infinite depth). For the converse, assume  $M$  is a three-dimensional module such that  $\mathbf{K}_1(M)$  has positive depth. Since  $\mathbf{K}_1(M^{\text{unm}})$  is a submodule of the latter by Lemma 2.6, it too has positive depth, and so we may assume from the start that  $M$  is unmixed. By assumption, we can find a parameter  $x$  which is  $\mathbf{K}_1(M)$ -regular. By unmixedness, it is also  $M$ -regular. Put  $\bar{R} := R/xR$  and  $\bar{M} := M/xM$ . From the short exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow \bar{M} \rightarrow 0$  we get by Proposition 2.3 and Grothendieck vanishing, a long exact sequence

$$0 = \mathbf{K}(\bar{M}) \rightarrow \mathbf{K}(M) \xrightarrow{x} \mathbf{K}(M) \rightarrow \mathbf{K}_1(\bar{M}) \rightarrow \mathbf{K}_1(M) \xrightarrow{x} \mathbf{K}_1(M)$$

By assumption the latter map is injective, showing that  $\mathbf{K}_1(\bar{M}) \cong \mathbf{K}(M)/x\mathbf{K}(M)$ . By (2.2.b), we have  $\mathbf{K}_1(\bar{M}) = \mathbf{K}_0^{\bar{R}}(\bar{M})$ , and by Corollary 2.7, this is an MCM over  $\bar{R}$ , that is to say, has depth two. Hence  $\mathbf{K}(M)$  has depth three, whence is an MCM over  $R$ .  $\square$

2.10. *Remark.* Our argument actually shows that if  $R$  is a  $d$ -dimensional complete local ring and  $M$  a  $d$ -dimensional  $R$ -module such that  $\mathbf{K}_1(M)$  has positive depth, then  $\mathbf{K}(M)$  has depth at least three.

### 3. THE FROBENIUS TRANSFORM OF A MODULE

In this section,  $(R, \mathfrak{m})$  will always denote a Noetherian local ring of characteristic  $p > 0$ , with residue field  $k$ . Moreover,  $q$  will always denote some power of  $p$ . We define its  $q$ -th degree of imperfection as  $\iota_q(R) := (\mathbf{F}_* k : k)$ , which is the same as the (vector space) degree of  $k$  over its subfield  $k^q$  of all  $q$ -th powers, and also the same as the degree of the field of  $q$ -th roots  $k^{1/q}$  over  $k$ . In particular,  $k$  is perfect if and only if  $\iota_p(R) = 1$ . A standing assumption, moreover, will be that  $R$  is  $F$ -finite, meaning that the Frobenius is a finite morphism. In particular, all degrees of imperfection  $\iota_q(R)$  will be finite, and the converse holds when  $R$  is complete.

We defined in the introduction the Frobenius transform functor  $\mathbf{F}_{q*}$  on the category of  $R$ -modules (the  $F$ -finiteness assumption implies that  $\mathbf{F}_* M$  is again finitely generated). Whenever  $q$  is clear from the context, we just denote it by  $\mathbf{F}_*$ . (A note of caution, do not confuse this with the Peskine-Szpiro Frobenius functor given by  $\mathfrak{F}(M) := M \otimes_R \mathbf{F}_* R$ .) Recall that  $\mathbf{F}_* M$  is the  $R$ -module whose elements are denoted by  $*m$ , for  $m \in M$ , and with the scalar action of  $R$  given by  $r*m := *r^p m$ . An easy, but very important property of the Frobenius transform is its exactness.

3.1. **Lemma.** *Given a multiplicative set  $\Sigma$ , we have  $\mathbf{F}_*(\Sigma^{-1}M) \cong \Sigma^{-1}(\mathbf{F}_*M)$ .*

*Proof.* Any element of  $\mathbf{F}_*(\Sigma^{-1}M)$  is of the form  $*(\frac{m}{s})$ , with  $m \in M$  and  $s \in \Sigma$ . To it, we let correspond the element  $\frac{*(s^{p-1}m)}{s}$  in  $\Sigma^{-1}(\mathbf{F}_*M)$ . To see that this is well defined, suppose  $\frac{m}{s} = \frac{\tilde{m}}{\tilde{s}}$  in  $\Sigma^{-1}M$ , for some  $\tilde{m} \in M$  and  $\tilde{s} \in \Sigma$ . Hence,  $\tilde{s}tm = st\tilde{m}$  in  $M$ , for some  $t \in \Sigma$ . Multiplying both sides with  $(s\tilde{s}t)^{p-1}$ , we get  $s^{p-1}\tilde{s}^p t^p m = s^p \tilde{s}^{p-1} t^p \tilde{m}$  in  $M$ , whence  $\tilde{s}t*(\tilde{s}^{p-1}m) = st*(s^{p-1}\tilde{m})$  in  $\mathbf{F}_*M$ , showing that  $\frac{*(s^{p-1}m)}{s} = \frac{*(\tilde{s}^{p-1}\tilde{m})}{\tilde{s}}$  in  $\Sigma^{-1}(\mathbf{F}_*M)$ . The converse is defined by sending  $\frac{*n}{t}$ , with  $n \in M$  and  $t \in \Sigma$  to  $*(\frac{n}{t^p})$ , which again, by a similar argument, is well-defined. Moreover, the compositions  $*(\frac{m}{s}) \mapsto \frac{*(s^{p-1}m)}{s} \mapsto *( \frac{s^{p-1}m}{s^p} ) = *( \frac{m}{s} )$  and  $\frac{*n}{t} \mapsto *( \frac{n}{t^p} ) \mapsto \frac{*(t^{p(p-1)}n)}{t^p} = \frac{t^{p-1}*n}{t^p} = \frac{*n}{t}$  are the identity, showing that these maps are each others inverse. We leave it to the reader to verify that these maps are  $R$ -linear, and therefore give the desired isomorphism.  $\square$

3.2. **Theorem.** *For each  $R$ -module  $M$  and each  $i \geq 0$ , we have a canonical isomorphism*

$$\mathbf{F}_* H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}}^i(\mathbf{F}_* M).$$

*Proof.* Let  $(x_1, \dots, x_d)$  be a system of parameters in  $R$  and let  $C_M^\bullet$  be the corresponding Čech complex on  $M$ , that is to say, for each  $i$ , the module  $C_M^i$  is the direct sum of all localizations  $M_z$  where  $z$  runs over all products of  $i$  distinct elements from the system of parameters, and the differential  $C_M^i \rightarrow C_M^{i+1}$  is given by the natural localization maps, up to a sign (for details, see [1, §3.5]). In particular,  $C_M^\bullet \cong C_R^\bullet \otimes_R M$ , and the homology of this complex is the local cohomology  $H_\bullet(C_M^\bullet) = H_\bullet^\bullet(M)$ . By Lemma 3.1, we have an isomorphism of complexes  $\mathbf{F}_*(C_M^\bullet) \cong C_{\mathbf{F}_*M}^\bullet$ . Taking homology and using that  $\mathbf{F}_*$ , being exact, commutes with homology, we get

$$\mathbf{F}_*(H_{\mathfrak{m}}^\bullet(M)) = \mathbf{F}_*(H_\bullet(C_M^\bullet)) \cong H_\bullet(\mathbf{F}_*C_M^\bullet) \cong H_\bullet(C_{\mathbf{F}_*M}^\bullet) = H_{\mathfrak{m}}^\bullet(\mathbf{F}_*M).$$

□

Immediate from this, we get:

**3.3. Corollary.** *If  $M$  is an MCM, then so is its Frobenius transform  $\mathbf{F}_*M$ .* □

**3.4. Proposition.** *If  $M$  has finite length, then  $\ell(\mathbf{F}_*M) = \iota \cdot \ell(M)$ , where  $\iota := \iota_q(R)$ .*

*Proof.* We want to show by induction on  $l := \ell(M)$  that  $\mathbf{F}_*M$  has length  $l\iota$ . When  $l = 1$ , then  $M \cong k$ , and the result holds by definition. For  $l > 1$ , choose an exact sequence  $0 \rightarrow N \rightarrow M \rightarrow k \rightarrow 0$  with  $\ell(N) = l - 1$ . Since  $\mathbf{F}_*$  is an exact functor, we get an exact sequence  $0 \rightarrow \mathbf{F}_*N \rightarrow \mathbf{F}_*M \rightarrow \mathbf{F}_*k \rightarrow 0$ , showing that  $\ell(\mathbf{F}_*M) = \ell(\mathbf{F}_*N) + \ell(\mathbf{F}_*k) = (l - 1)\iota + \iota = l\iota$ , as we wanted to show. □

**3.5. Corollary.** *For any module  $M$  of finite length  $l$  over a local ring  $R$  with perfect residue field  $k$ , we have  $\mathbf{F}_*^n M \cong k^l$ , for all  $n \gg 0$ .*

*Proof.* Since  $\mathfrak{m}^l$  annihilates  $M$ , an easy induction argument yields that  $\mathfrak{m}$  annihilates  $\mathbf{F}_*^l M$ . Hence  $\mathbf{F}_*^l M$  is a vector space, of length  $l$  by Proposition 3.4. □

Recall that  $\ell_0(M) := \ell(H_{\mathfrak{m}}^0(M))$ , which is therefore the largest length of an Artinian submodule of  $M$ . We can now give a more general form of (2):

**3.6. Corollary.** *With  $\iota := \iota_q(R)$ , we have, for any finitely generated  $R$ -module,*

$$(6) \quad \ell_0(\mathbf{F}_*M) = \iota \ell_0(M).$$

*Proof.* Let  $H := H_{\mathfrak{m}}^0(M)$  be the maximal Artinian submodule of  $M$ . It is not hard to see that  $\mathbf{F}_*H$  is then the maximal Artinian submodule of  $\mathbf{F}_*M$ . The result follows, since  $\ell(\mathbf{F}_*H) = \iota \ell(H)$  by Proposition 3.4. □

#### 4. THE FROBENIUS TRANSFORM OF A PSEUDO-CANONICAL MODULE

We will interpret  $m$ -tuples  $\mathbf{x}$  as row vectors and so, their image under an  $m \times n$ -matrix  $\mathbb{A}$  is given by  $\mathbf{x}\mathbb{A}$ . For our purposes, it is more convenient to use the logicians' numbering, that is to say, we list the entries  $a_{ij}$  of  $\mathbb{A}$  from  $i = 0, \dots, m - 1$  and  $j = 0, \dots, n - 1$ .

Given a morphism  $f: M \rightarrow N$ , the induced morphism  $\mathbf{F}_*f: \mathbf{F}_*M \rightarrow \mathbf{F}_*N$  is given by the rule  $*m \mapsto *f(m)$ . Let  $r_M^\times$  be the endomorphism on  $M$  given by multiplication of a scalar  $r \in R$ . Note that  $\mathbf{F}_*(r_M^\times)$  is in general not multiplication with  $r$ , but, in fact

$$(\mathbf{F}_*(r_M^\times))^q = r_{\mathbf{F}_*M}^\times.$$

For any module  $M$ , we obtain a ring homomorphism<sup>2</sup>

$$\nabla_M: R \rightarrow \text{End}(\mathbf{F}_*M): r \mapsto \mathbf{F}_*(r_M^\times).$$

Given an  $m \times n$ -matrix  $\mathbb{A}$  and a module  $M$ , we get a morphism  $M^m \rightarrow M^n: \mathbf{m} \mapsto \mathbf{m}\mathbb{A}$ , which we will denote similarly by  $\mathbb{A}_M^\times$  (and when there is no possibility for confusion, we just write the matrix for the morphism it defines). However, for the converse, a morphism  $M^m \rightarrow M^n$  is given by a matrix with entries over the (possibly non-commutative) endomorphism ring  $\text{End}(M)$ . The Frobenius transform  $\mathbf{F}_*(\mathbb{A}_M^\times)$  is given by the rule  $\mathbf{m} \mapsto *\mathbf{m}\mathbb{A}$ , and its corresponding matrix has coefficients in  $\text{End}(\mathbf{F}_*M)$ , namely:

<sup>2</sup>Note that this is not a morphism of  $R$ -algebras; to make it into one, we should instead consider the morphism  $\mathbf{F}_*R \rightarrow \text{End}(\mathbf{F}_*M): *r \mapsto \mathbf{F}_*(r_M^\times)$ , but this will be of lesser use to us.



**4.1. Lemma.** *Given an  $m \times n$ -matrix  $\mathbb{A}$  over  $R$ , we will write  $\mathbb{A}^{\nabla_M}$  for the matrix over  $\text{End}(\mathbf{F}_*M)$  obtained from  $\mathbb{A}$  by applying  $\nabla_M: R \rightarrow \text{End}(\mathbf{F}_*M)$  to each of its entries. Then*

$$\mathbf{F}_*(\mathbb{A}_M^\times) = (\mathbb{A}^{\nabla_M})_{\mathbf{F}_*M}^\times. \quad \square$$

**Quasi-symmetric matrices.** We need some terminology from linear algebra. The  $n \times n$ -identity matrix will be denoted by  $\mathbb{I}_n$ , or just  $\mathbb{I}$ , if its size is clear from the context. We also need the  $n \times n$ -exchange matrix  $\mathbb{J}_n$  (or just  $\mathbb{J}$ ), which is the matrix with ones on the counterdiagonal (=SE/NW diagonal) and zeros elsewhere. Note that  $\mathbb{J}^2 = \mathbb{I}$ , and in particular,  $\mathbb{J}$  is its own inverse. Fix  $n$  and let  $\mathbb{A} = (a_{ij})$  be a square  $n \times n$ -matrix. Recall that  $\mathbb{A}$  is called *symmetric* if it is equal to its own transpose  $\mathbb{A}^T$ ; in matrix entries, this means  $a_{ij} = a_{ji}$ . We call  $\mathbb{A}$  *persymmetric*, if it is symmetric around its counterdiagonal. This is equivalent with  $\mathbb{A}\mathbb{J} = \mathbb{J}\mathbb{A}^T$ . In terms of its entries, this means,

$$(7) \quad a_{ij} = a_{n-1-j, n-1-i}.$$

**The Frobenius transform over a regular local ring.** Let  $S$  be a  $d$ -dimensional complete, regular local ring of characteristic  $p$  with perfect residue field  $k$ . By Kunz's theorem,  $\mathbf{F}_*S$  is a free  $S$ -module of rank  $n := q^d$ . More concretely,  $S \cong k[[\mathbf{x}]]$ , with  $\mathbf{x} = (x_0, \dots, x_{d-1})$ . For  $a = 0, \dots, q^d - 1$ , let  $\hat{a}_k$  be its  $q$ -adic digits, that is to say,  $a = \sum \hat{a}_k q^k$  is the  $q$ -adic expansion of  $a$ , with  $0 \leq \hat{a}_k < q$ , and put

$$\mathbf{m}_a := x_0^{\hat{a}_0} x_1^{\hat{a}_1} \cdots x_{d-1}^{\hat{a}_{d-1}}$$

The  $*\mathbf{m}_a$ , for  $a = 0, \dots, q^d - 1$ , then form a basis of  $\mathbf{F}_*S$  over  $S$ , called the standard basis and we will consider all our matrices with respect to this basis (in the given order). Given  $s \in S$ , we want to describe the endomorphism  $\nabla_S(s) = \mathbf{F}_*(s_S^\times)$  on  $\mathbf{F}_*S$ . Let us denote its matrix with respect to the standard basis by  $\mathbb{D}_S(s)$  or just  $\mathbb{D}(s)$ .

**4.2. Proposition.** *Each matrix  $\mathbb{D}(s)$  is persymmetric.*

*Proof.* Any linear combination of persymmetric matrices is again persymmetric. Since  $S$  is generated by its monomials, it suffices therefore to show the claim for  $s = \mathbf{x}^\gamma$  a monomial, with  $\gamma = (c_0, \dots, c_{d-1}) \in \mathbb{N}^d$ . Let  $s_{ab}$ , for  $a, b < q^d$ , be the entries of  $\mathbb{D}(s)$ . For an arbitrary integer  $c$ , let  $\mathbf{q}(c)$  and  $\mathbf{r}(c)$  be its respective quotient and remainder after division by  $q$ , so that  $c = q\mathbf{q}(c) + \mathbf{r}(c)$ . Define a permutation, denoted again  $s$ , on the indices by the rule

$$s(a) := \sum_{k=0}^{d-1} \mathbf{r}(c_k + \hat{a}_k) q^k$$

Hence,

$$\begin{aligned} \mathbf{F}_*(s_S^\times)(\mathbf{m}_a) &= *\mathbf{x}^\gamma \mathbf{m}_a = \prod_{k=0}^{d-1} x_k^{\mathbf{q}(c_k + \hat{a}_k)} * \prod_{k=0}^{d-1} x_k^{\mathbf{r}(c_k + \hat{a}_k)} \\ &= \prod_{k=0}^{d-1} x_k^{\mathbf{q}(c_k + \hat{a}_k)} * \mathbf{m}_{s(a)} \end{aligned}$$

Hence  $s_{ab}$  is zero, unless  $b = s(a)$ , in which case it is equal to

$$(8) \quad s_{ab} = \prod_{k=0}^{d-1} x_k^{\mathbf{q}(c_k + \hat{a}_k)}.$$



We need to verify identity (7). From

$$q^d - 1 - a = \sum_{k=0}^{d-1} (q - 1 - \mathring{a}_k) q^k$$

we read off its  $q$ -adic digits as  $q - 1 - \mathring{a}_k$ . Hence, applied to  $b$ , we get

$$s(q^d - 1 - b) = \sum_{k=0}^{d-1} \mathbf{r}(c_k + q - 1 - \mathring{b}_k) q^k$$

and this equal to  $q^d - 1 - a$  if and only if  $\mathbf{r}(c_k + q - 1 - \mathring{b}_k) = q - 1 - \mathring{a}_k$ . The latter equality is the same as the equivalence  $c_k - \mathring{b}_k \equiv -\mathring{a}_k \pmod{q}$  whence  $c_k + \mathring{a}_k \equiv \mathring{b}_k \pmod{q}$ , which in turn is equivalent with  $s(a) = b$ . Therefore, if  $a \neq s(b)$ , then  $s_{q^d-1-b, q^d-1-a} = 0$ , and so we only need to calculate it in the case that  $a = s(b)$ . In that case,

$$(9) \quad s_{q^d-1-b, q^d-1-a} = \prod_{k=0}^{d-1} x_k^{\mathbf{q}(c_k + q - 1 - \mathring{b}_k)}.$$

Since  $c_k + \mathring{a}_k \equiv \mathring{b}_k \pmod{q}$ , we can write

$$(10) \quad c_k + \mathring{a}_k = qv_k + \mathring{b}_k,$$

for some  $v_k$ . Since  $\mathring{b}_k < q$ , we see that  $v_k = \mathbf{q}(c_k + \mathring{a}_k)$ . On the other hand, using (10), we see that  $c_k + q - 1 - \mathring{b}_k = qv_k + q - 1 - \mathring{a}_k$ , and since  $q - 1 - \mathring{a}_k < q$ , this shows that  $v_k = \mathbf{q}(c_k + q - 1 - \mathring{b}_k)$ , so that (8) and (9) are the same, as we wanted to show.  $\square$

Let  $E$  be the injective hull of  $k$ .

**4.3. Proposition.** *We have a canonical isomorphism  $\mathbf{F}_*E \cong E \otimes \mathbf{F}_*S$ .*

*Proof.* Since  $S$  is in particular Gorenstein, whence equal to its own canonical module,  $H_{\mathbf{m}}^d(S) \cong E$  by Grothendieck duality. Using Theorem 3.2, we get canonical isomorphisms

$$\mathbf{F}_*E = \mathbf{F}_*H_{\mathbf{m}}^d(S) = H_{\mathbf{m}}^d(\mathbf{F}_*S) \cong H_{\mathbf{m}}^d(S) \otimes \mathbf{F}_*S \cong E \otimes \mathbf{F}_*S.$$

In fact, representing  $E = H_{\mathbf{m}}^d(S)$  as the cokernel of the Čech complex  $C_S^{d-1} \rightarrow C_S^d = S_x$ , where  $x = x_1 \cdots x_d$  for a fixed system of parameters  $(x_1, \dots, x_d)$ , and writing  $[\frac{a}{x^n}]$  for the image of  $a/x^n$  in  $H_{\mathbf{m}}^d(S)$ , we can trace the above isomorphism explicitly, and show that it is given by

$$(11) \quad E \otimes \mathbf{F}_*S \rightarrow \mathbf{F}_*E: [\frac{a}{x^n}] \otimes *r \mapsto *[\frac{ra^p}{x^{np}}]$$

To define the converse, observe that any element in  $E$  is of the form  $[\frac{a}{x^{pn}}]$ , by multiplying numerator and denominator with a suitable power of  $x$ . The corresponding element  $*[\frac{a}{x^{pn}}]$  is then sent under the inverse isomorphism to  $[\frac{1}{x^n}] \otimes *a$ .  $\square$

**4.4. Remark.** Note that we only used the fact that  $E \cong H_{\mathbf{m}}^d(S)$ , and so the proof works in fact for any quasi-Gorenstein ring  $S$ .

Since  $\mathbf{F}_*S \cong S^{q^d}$ , we get in fact  $\mathbf{F}_*E \cong E^{q^d}$ . In particular, since  $\text{End}(E) = S$ , the Frobenius transform of multiplication with an element  $s \in S$  on  $E$ , that is to say,  $\mathbf{F}_*(s_E^\times)$ , is given by a  $q^d \times q^d$ -matrix with coefficients in  $S$ , which we will denote by  $\mathbb{D}_E(s)$ .

**4.5. Lemma.** *For any element  $s \in S$ , we have an equality of matrices  $\mathbb{D}_E(s) = \mathbb{D}_S(s)$ .*

*Proof.* Let  $s_{ab}$  be the entries of  $\mathbb{D}_S(s)$ . The  $(a, b)$ -th entry of  $\mathbb{D}_E(s)$  is the endomorphism given by the composition

$$E \xrightarrow{i_a} E \otimes \mathbf{F}_* S \cong \mathbf{F}_* E \mathbf{F}_* (s_E^\times) qq \rightarrow \mathbf{F}_* E \cong E \otimes \mathbf{F}_* S \xrightarrow{\pi_b} E$$

where the isomorphisms are given by (11), and  $i_a$  (respectively,  $\pi_b$ ) is the base change of the  $a$ -th embedding (respectively  $b$ -th projection map) of  $S$  into  $\mathbf{F}_* S$  (respectively, of  $\mathbf{F}_* S$  onto  $S$ ). Since  $\text{End}(E) = S$ , this endomorphism is then given by multiplication with an element, and we need to show that this is  $s_{ab}$ . To this end, take  $z \in E$ . In the notation of the previous proof, it is of the form  $z = [\frac{v}{x^n}]$ , with  $v \in S$ ,  $n \in \mathbb{N}$ , and  $x := x_0 \cdots x_{d-1}$ . Its image under the above composition is

$$z \xrightarrow{i_a} z \otimes * \mathbf{m}_a \xrightarrow{(11)} * [\frac{v^q \mathbf{m}_a}{x^{nq}}] \xrightarrow{\mathbf{F}_* (s_E^\times)} * [\frac{v^q s \mathbf{m}_a}{x^{nq}}] \xrightarrow{(11)} [\frac{v}{x^n}] \otimes * s \mathbf{m}_a \xrightarrow{\pi_b} s_{ab} z$$

since  $* s \mathbf{m}_a = \sum_b s_{ab} * \mathbf{m}_b$ , proving the claim.  $\square$

To prove (1), we temporarily introduce the following functor on the category of finitely generated modules

$$(12) \quad \mathbf{T}(M) := (\mathbf{F}_*(M^\vee))^\vee$$

**4.6. Theorem.** *Over a complete local ring  $R$ , we have  $\mathbf{F}_*(M) \cong \mathbf{T}(M)$ , for any  $R$ -module  $M$ .*

*Proof.* Assume first that  $R$  is regular of dimension  $d$ , so that  $\mathbf{F}_* R \cong R^{q^d}$  by Kunz's Theorem. Using Proposition 4.3, we get

$$\mathbf{T}(R) = (\mathbf{F}_* E)^\vee \cong (E^{q^d})^\vee = R^{q^d} \cong \mathbf{F}_* R.$$

For  $M$  an arbitrary finitely generated  $R$ -module, we can represent it as the cokernel of a (square) matrix  $\mathbb{A}$ , that is to say, by an exact sequence  $R^n \mathbb{A} qq \rightarrow R^n \rightarrow M \rightarrow 0$ . By Lemma 4.1, its Frobenius transform is

$$(13) \quad \mathbf{F}_* R^n \mathbb{A}^{\nabla_R} qq \rightarrow \mathbf{F}_* R^n \rightarrow \mathbf{F}_* M \rightarrow 0$$

Recall that  $\mathbb{A}^{\nabla_R}$  is an  $n \times n$ -matrix of  $(q^d \times q^d)$ -matrices: if the  $(i, j)$ -th entry of  $\mathbb{A}$  is  $a_{ij}$ , then the  $(i, j)$ -th entry of  $(\mathbb{A}^T)^{\nabla_R}$  is the matrix  $\mathbb{D}_R(a_{ji})$ . On the other hand, by Matlis duality  $0 \rightarrow M^\vee \rightarrow E^n \rightarrow E^n$  is given by the transpose  $\mathbb{A}^T$ . Taking again the Frobenius transform and using Lemma 4.1, we get an exact sequence

$$0 \rightarrow \mathbf{F}_*(M^\vee) \rightarrow (\mathbf{F}_* E)^n \xrightarrow{(\mathbb{A}^T)^{\nabla_E}} (\mathbf{F}_* E)^n$$

and taking one more time Matlis duals, an exact sequence

$$(14) \quad \mathbf{T}(R)^n ((\mathbb{A}^T)^{\nabla_E})^T qq \rightarrow \mathbf{T}(R)^n \rightarrow \mathbf{T}(M) \rightarrow 0$$

Let  $\mathbb{P}$  be the matrix  $\text{diag}(\mathbb{J}, \dots, \mathbb{J})$ , that is to say, the diagonal  $n \times n$ -matrix with diagonal elements the  $q^d$ -th exchange matrix  $\mathbb{J}$ . Since each  $\mathbb{D}_E(a_{ji})$  is persymmetric by Proposition 4.2 and equal to  $\mathbb{D}_R(a_{ji})$  by Lemma 4.5, one easily verifies the following matrix relation

$$\mathbb{P} \cdot (\mathbb{A}^T)^{\nabla_E} \cdot \mathbb{P}^{-1} = (\mathbb{A}^{\nabla_R})^T$$

Hence, as  $\mathbf{T}(R) \cong \mathbf{F}_* R$  and the matrices in (13) and (14) are conjugate, their cokernels are isomorphic, as we needed to show.

For  $R$  arbitrary, by Cohen's structure theorem, we can find a regular local ring  $S$  and a surjection  $S \rightarrow R$ . The functor  $\mathbf{F}_*(\cdot)$  on the category of  $R$ -modules remains the same, if we consider them instead as  $S$ -modules. By Proposition 2.2, the same is true for Matlis duality, whence also for the composite functor  $\mathbf{T}(\cdot)$ . Hence, as  $S$ -modules, we have an isomorphism  $\mathbf{F}_*M \cong \mathbf{T}(M)$ , which therefore is also an isomorphism over  $R$ .  $\square$

4.7. **Proof of (1).** For all  $i$ , we have

$$\begin{aligned} \mathbf{K}_i(\mathbf{F}_*M) &\stackrel{2.1}{\cong} (\mathbf{H}_m^{d-i}(\mathbf{F}_*M))^\vee \stackrel{3.2}{\cong} (\mathbf{F}_* \mathbf{H}_m^{d-i}(M))^\vee \\ &\stackrel{2.1}{\cong} (\mathbf{F}_*(\mathbf{K}_i(M)^\vee))^\vee \stackrel{(12)}{=} \mathbf{T}(\mathbf{K}_i(M)) \stackrel{4.6}{\cong} \mathbf{F}_*\mathbf{K}_i(M). \quad \square \end{aligned}$$

We also derive the following dual version of Kunz's theorem:

4.8. **Theorem.** *A complete local ring  $R$  with perfect residue field  $k$  is regular if and only if  $\mathbf{F}_*E$  is injective, where  $E$  is the injective hull of  $k$ .*

*Proof.* One direction is immediate from Proposition 4.3, so assume  $\mathbf{F}_*E$  is injective, necessarily of the form  $E^n$ . Theorem 4.6 then yields  $\mathbf{F}_*R \cong \mathbf{T}(R) = (\mathbf{F}_*E)^\vee = R^n$ , so that by Kunz's theorem,  $R$  is regular.  $\square$

**F-trivialization.** We have already seen in Corollary 3.5 that Frobenius transforms trivialize modules of finite length. We now conjecture that a weaker version of this is true in general (we formulate it only for top dimension, as this is the only case we need, but presumably this holds for any non-simple module):

4.9. **Conjecture.** *A complete  $d$ -dimensional local ring  $R$  in positive characteristic is F-trivializing, meaning that for every  $d$ -dimensional  $R$ -module  $M$ , there is some  $n$  such that  $\mathbf{F}_*^n M$  is decomposable.*

4.10. **Theorem.** *An F-trivializing complete three-dimensional local ring  $R$  admits an MCM.*

*Proof.* With  $h$  as defined in the introduction (after (2)), suppose  $R$  does not admit an MCM. Hence  $h(M) > 0$ , for all three-dimensional modules  $M$ . Choose  $M$  with  $h(M)$  minimal. By assumption, we can find  $q = p^n$  and a decomposition  $\mathbf{F}_*M = Q \oplus N$ . Using (3) and the additivity of  $h$ , we get  $h(M) = h(\mathbf{F}_*M) = h(Q) + h(N)$ . But the latter sum violates the minimality of  $h(M)$ , contradiction.  $\square$

To verify F-trivialization, we must check all  $d$ -dimensional modules, and I do not even know whether this is true for regular local rings of dimension  $d \geq 2$ . However, less is needed for the proof: let us call a class  $\mathfrak{C}$  of  $d$ -dimensional modules an *F-net*, if it is closed under Frobenius transforms and direct summands. We leave it to the reader to check that the class of unmixed modules is an F-net. We say that  $R$  is *weakly F-trivializing*, if there exists an F-net  $\mathfrak{C}$  such that for each  $M \in \mathfrak{C}$ , there exists  $n \in \mathbb{N}$  for which  $\mathbf{F}_*^n M$  is decomposable. We may now weaken the assumption in Theorem 4.10 to being weakly F-trivializing: indeed, in the proof, just take the minimal  $h(M)$  for  $M$  in the F-net. For a given module  $Q$ , let  $\mathfrak{F}_Q$  be the collection of direct summands of its Frobenius transforms  $\mathbf{F}_*^n Q$ . If  $Q$  is unmixed, then so is any module in  $\mathfrak{F}_Q$ , and so this is an example of an F-net; in fact, it is the smallest F-net generated by  $Q$ . A regular local ring is now easily seen to be weakly F-trivializing: indeed,  $\mathfrak{F}_S$  is just the class of free modules, and so is clearly F-trivializing. We may restate this now as follows:

**4.11. Corollary.** *If  $R$  is a complete three-dimensional local ring admitting an unmixed module  $Q$  such that any direct summand of a Frobenius transform of  $Q$  has itself a Frobenius transform that is decomposable, then  $R$  admits an MCM.*  $\square$

In the examples below, the structure of the F-net  $\mathfrak{F}_R$  appears already a quite intriguing invariant of  $R$ .

## 5. EXAMPLES

We will present most examples without the calculations (they become tedious in higher dimensions, but explaining the methods to tackle these would be the topic of a paper on its own). Unfortunately, all examples are Cohen-Macaulay, and so their weak F-splitness is rather a moot point.

**5.1. Example** (Regular local rings). Let  $(S, \mathfrak{m})$  be the power series ring in  $d$  variables over a perfect field  $k$  of characteristic 3. If  $d = 2$ , we have

$$\mathbf{F}_*S \cong S^9 \quad \text{and} \quad \mathbf{F}_*\mathfrak{m} \cong \mathfrak{m} \oplus S^8$$

showing that the F-net  $\mathfrak{F}_{\mathfrak{m}}$  consists of direct sums of copies of  $R$  and  $\mathfrak{m}$ . Similarly, one can show that  $\mathbf{F}_*\mathfrak{m}^2 \cong \mathfrak{m}S^3 \oplus S^6$  and  $\mathbf{F}_*\mathfrak{m}^3 \cong \mathfrak{m}S^6 \oplus S^3$ . However, the structure of  $\mathbf{F}_*\mathfrak{m}^4$  seems no longer to follow this pattern as it contains a free summand of rank four.

For  $d = 3$ , after some lengthy calculations, we find  $\mathbf{F}_*\mathfrak{m} \cong \mathfrak{m} \oplus S^{26}$ . It seems therefore reasonable to postulate the same in arbitrary dimension and characteristic

$$\mathbf{F}_*\mathfrak{m} \stackrel{?}{\cong} \mathfrak{m} \oplus S^{p^d-1}.$$

**5.2. Example** (Curves). Let  $(R, \mathfrak{m})$  be the local ring at the origin of the cusp  $x^2 = y^3$  with  $p = 3$ . Note that  $R$  is not F-pure (for in dimension one this is equivalent with being regular). Indeed, one verifies that

$$\mathbf{F}_*R \cong \mathbf{F}_*\mathfrak{m} \cong \mathfrak{m}R^3.$$

In particular,  $\mathfrak{m}$  is F-split whence  $R$  is weakly F-split. Moreover, the F-net  $\mathfrak{F}_R$  consists of all direct sums of copies of  $R$  and  $\mathfrak{m}$ , and  $R$  is F-trivializing for this F-net, that is to say,  $R$  is weakly F-trivializing.

Take instead the local ring of the curve  $x^3 = y^4$  in characteristic  $p = 5$ . We have

$$\mathbf{F}_*R \cong (x, y^2)R \oplus \mathfrak{m}^2R^4 \quad \text{and} \quad \mathbf{F}_*\mathfrak{m}^2 \cong \mathbf{F}_*(x, y^2)R \cong \mathfrak{m}^2R^5.$$

In particular,  $R$  is weakly F-split witnessed by the F-split module  $\mathfrak{m}^2$ , and also weakly F-trivializing, witnessed by the F-net  $\mathfrak{F}_R$ , which is generated by the three indecomposables  $R$ ,  $(x, y^2)R$  and  $\mathfrak{m}^2$ .

**5.3. Example** (Surfaces). Let  $R$  be the local ring at the origin of the surface  $x^3 = y^2z$ . By Federer's criterion [2], one can see that  $R$  is not F-split when  $p = 3$ . In fact, we have

$$\mathbf{F}_*R \cong (x, y)^2R^3 \oplus (x^2, y)R^3 \oplus (x^2, yz)R^3$$

When  $p = 5$ , we can show that  $(x^2, y)R$  is a direct summand of  $\mathbf{F}_*R$  and also a direct summand of its own Frobenius transform  $\mathbf{F}_*(x^2, y)R$ , i.e., F-split, showing that  $R$  is weakly F-split.

For  $p = 7$ , we look at the surface  $x^2 = y^4z^5$ . In [8], we introduced the notion of F-integral closure (=collection of fractions  $f/g$  some  $p^n$ -th power of which lies in  $R$ ), and showed that they often are already MCM's. In this case, its F-integral closure

$R' := R[\frac{x}{yz^2}]$  is indeed an MCM, but it is also F-split:  $R'$  is a direct summand of  $\mathbf{F}_* R'$ .

## REFERENCES

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